

On the correspondence between mirror-twisted sectors for N=2 supersymmetric vertex operator superalgebras of the form $V \otimes V$ and N=1 Ramond sectors of V

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Abstract Using recent results of the author along with Vander Werf, we present the classification and construction of mirror-twisted modules for N=2 supersymmetric vertex operator superalgebras of the form $V \otimes V$ for the signed transposition mirror map automorphism. In particular, we show that the category of such mirror-twisted sectors for $V \otimes V$ is isomorphic to the category of N=1 Ramond sectors for V .

1 Introduction

In [B4], [B5], the author studied twisted modules for N=2 supersymmetric vertex operator superalgebras (N=2 VOSAs) for finite order VOSA automorphisms arising from automorphisms of the N=2 Neveu-Schwarz algebra of N=2 infinitesimal superconformal transformations. Among such automorphisms is the mirror map. In [B4], mirror maps were given for N=2 VOSAs of the form $V \otimes V$ where V is an N=1 supersymmetric VOSA of the form $V_L \otimes V_{fer}$, where V_L is a rank d lattice VOSA or the d free boson vertex operator algebra and V_{fer} is the d free fermion VOSA. In particular, we showed that one of the mirror maps for such an N=2 VOSA, $V \otimes V$, is given by the signed transposition map

$$\tilde{\kappa} = \begin{pmatrix} 1 & 2 \end{pmatrix} : V \otimes V \longrightarrow V \otimes V, \quad u \otimes v \mapsto (-1)^{|u||v|} v \otimes u \quad (1)$$

where $|v| = j \bmod 2$ for $v \in V^{(j)}$, with the \mathbb{Z}_2 -grading of V given by $V = V^{(0)} \oplus V^{(1)}$.

In [B6] and [BV2], the author along with Vander Werf constructed and classified the cyclic permutation-twisted $V^{\otimes k}$ -modules, where V is any VOSA and k is a positive integer. For k even, this classification is in terms of parity-twisted V -modules where the parity automorphism of a VOSA is the map

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$$\sigma : V \longrightarrow V, \quad v \mapsto (-1)^{|v|}v. \quad (2)$$

In this note, we apply the results of [BV2] to the setting of the mirror map (1) acting on an N=2 supersymmetric VOSA of the form $V \otimes V$, to show that the category of $\bar{\kappa}$ -twisted $(V \otimes V)$ -modules is isomorphic to the category of σ -twisted V -modules, which are the N=1 Ramond sectors for the N=1 supersymmetric VOSA, V . This classification also provides an explicit construction of these modules.

In particular, our result shows that if a representation M_σ of the N=1 Ramond algebra is also a parity-twisted modules for a VOSA V , where $V \otimes V$ is N=2 supersymmetric, then M_σ is also naturally a representation of the mirror-twisted N=2 Neveu-Schwarz algebra. These results can be used to calculate the graded dimensions for one module in terms of the graded dimensions for the other as shown in Corollary 2 below. Note that for our results, we do not need to make any assumptions about, for instance, the values of the central charge, the complete reducibility of the representations, or the rationality of the VOSAs.

Certain representations of the N=1 Ramond algebra and related VOSA constructions have previously been studied in, e.g., [FQS], [GKO], [FFR], [L], [S], [IK1], [Mi], [AM]. Certain representations of the mirror-twisted N=2 Neveu-Schwarz algebra have previously been studied in, e.g., [BFK], [Dob], [Ma], [K], [DG2], [G], [IK2], [LSZ]. In particular, the relationship between characters of certain modules for the N=1 Ramond algebra and certain modules for the mirror-twisted N=2 Neveu-Schwarz algebra had previously been observed. Our explicit isomorphism between mirror-twisted sectors for $V \otimes V$ and N=1 Ramond sectors for V , gives a constructive and overarching explanation of this phenomenon through the theory of VOSAs.

2 The notions of VOSA and twisted module

Following the notation of [B6], [BV2], we recall the notion of VOSA and the notions of weak, weak admissible and ordinary g -twisted V -module for a VOSA, V , and an automorphism g of V of finite order.

Let x, x_0, x_1, x_2 , denote commuting independent formal variables. Let $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$. Expressions such as $(x_1 - x_2)^n$ for $n \in \mathbb{C}$ are to be understood as formal power series expansions in nonnegative integral powers of the second variable.

Definition 1. A *vertex operator superalgebra* is a $\frac{1}{2}\mathbb{Z}$ -graded (by weight) vector space $V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$, satisfying $\dim V_n < \infty$ and $V_n = 0$ for n sufficiently negative, that is also \mathbb{Z}_2 -graded by *sign*, $V = V^{(0)} \oplus V^{(1)}$, with $V^{(j)} = \bigoplus_{n \in \mathbb{Z} + \frac{j}{2}} V_n$, and equipped with a linear map

$$V \longrightarrow (\text{End } V)[[x, x^{-1}]], \quad v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \quad (3)$$

and with two distinguished vectors $\mathbf{1} \in V_0$, (the *vacuum vector*) and $\omega \in V_2$ (the *conformal element*) satisfying the following conditions for $u, v \in V$: $u_n v = 0$ for n

sufficiently large; $Y(\mathbf{1}, x)v = v$; $Y(v, x)\mathbf{1} \in V[[x]]$, and $\lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v$;

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - (-1)^{|u||v|} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) \end{aligned} \quad (4)$$

(the *Jacobi identity*), where $|v| = j$ if $v \in V^{(j)}$ for $j \in \mathbb{Z}_2$; writing $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$, i.e., $L(n) = \omega_{n+1}$, for $n \in \mathbb{Z}$, then the $L(n)$ give a representation of the Virasoro algebra with central charge $c \in \mathbb{C}$ (the *central charge* of V); for $n \in \frac{1}{2}\mathbb{Z}$ and $v \in V_n$, then $L(0)v = nv = (\text{wt } v)v$; and the $L(-1)$ -*derivative property* holds: $\frac{d}{dx}Y(v, x) = Y(L(-1)v, x)$.

An *automorphism* of a VOSA, V , is a linear map g from V to itself, preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v, x)$ on V are compatible in the sense that $gY(v, x)g^{-1} = Y(gv, x)$, for $v \in V$. Then $gV_n \subset V_n$ for $n \in \frac{1}{2}\mathbb{Z}$.

Let \mathbb{Z}_+ denote the positive integers. If g has finite order, V is a direct sum of the eigenspaces V^j of g , i.e., $V = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} V^j$, where $k \in \mathbb{Z}_+$ is a period of g (i.e., $g^k = 1$) and $V^j = \{v \in V \mid gv = \eta^j v\}$, for η a fixed primitive k -th root of unity.

Definition 2. Let $(V, Y, \mathbf{1}, \omega)$ be a VOSA and g an automorphism of V of period $k \in \mathbb{Z}_+$. A *weak g -twisted V -module* is a vector space M equipped with a linear map

$$V \longrightarrow (\text{End } M)[[x^{1/k}, x^{-1/k}]], \quad v \mapsto Y^g(v, x) = \sum_{n \in \frac{1}{k}\mathbb{Z}} v_n^g x^{-n-1}, \quad (5)$$

with $v_n^g \in (\text{End } M)^{(|v|)}$, and satisfying the following conditions for $u, v \in V$ and $w \in M$: $v_n^g w = 0$ for n sufficiently large; $Y^g(\mathbf{1}, x)w = w$;

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y^g(u, x_1) Y^g(v, x_2) - (-1)^{|u||v|} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y^g(v, x_2) Y^g(u, x_1) \\ = x_2^{-1} \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} \delta \left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) Y^g(Y(g^j u, x_0)v, x_2) \end{aligned} \quad (6)$$

(the *twisted Jacobi identity*) where η is a fixed primitive k -th root of unity.

As a consequence of the definition, we have that $Y^g(v, x) = \sum_{n \in \mathbb{Z} + \frac{j}{k}} v_n^g x^{-n-1}$ for $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$, and for $v \in V$, we have $Y_g(gv, x) = \lim_{x^{1/k} \rightarrow \eta^{-1}x^{1/k}} Y^g(v, x)$. It also follows that writing $Y^g(\omega, x) = \sum_{n \in \mathbb{Z}} L^g(n)x^{-n-2}$, i.e., setting $L^g(n) = \omega_{n+1}^g$, for $n \in \mathbb{Z}$, then the $L^g(n)$ satisfy the relations for the Virasoro algebra with central charge c the central charge of V .

If we take $g = 1$, then we obtain the notion of weak V -module. The term “weak” means we are making no assumptions about a grading on M .

A *weak admissible g -twisted V -module* is a weak g -twisted V -module M which carries a $\frac{1}{2k}\mathbb{N}$ -grading $M = \bigoplus_{n \in \frac{1}{2k}\mathbb{N}} M(n)$, such that $v_m^g M(n) \subseteq M(n + \text{wt } v - m - 1)$

for homogeneous $v \in V$, $n \in \frac{1}{2k}\mathbb{N}$, and $m \in \frac{1}{k}\mathbb{Z}$. If $g = 1$, then a weak admissible g -twisted V -module is called a weak admissible V -module.

An (ordinary) g -twisted V -module is a weak g -twisted V -module M graded by \mathbb{C} induced by the spectrum of $L(0)$. That is, we have $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M \mid L(0)^g w = \lambda w\}$, for $L(0)^g = \omega_1^g$. Moreover we require that $\dim M_\lambda$ is finite and $M_{n/2k+\lambda} = 0$ for fixed λ and for all sufficiently small integers n . If $g = 1$, then a g -twisted V -module is a V -module.

3 The construction and classification of $(1\ 2\ \dots\ k)$ -twisted $V^{\otimes k}$ -modules

Let $V = (V, Y, \mathbf{1}, \omega)$ be a VOSA, and let k be a fixed positive integer. Then $V^{\otimes k}$ is also a VOSA, and the permutation group S_k acts naturally on $V^{\otimes k}$ as signed automorphisms, i.e., as a right action we have

$$(1\ 2\ \dots\ k) : V \otimes V \otimes \dots \otimes V \longrightarrow V \otimes V \otimes \dots \otimes V \quad (7)$$

$$v_1 \otimes v_2 \otimes \dots \otimes v_k \mapsto (-1)^{|v_1|(|v_2|+\dots+|v_k|)} v_2 \otimes v_3 \otimes \dots \otimes v_k \otimes v_1.$$

Let $g = (1\ 2\ \dots\ k)$. Below, we will recall the classification and construction of g -twisted $V^{\otimes k}$ -modules from [B6] and [BV2]. This construction is based on a certain operator $\Delta_k(x)$ first defined in [BDM], which we now recall.

Consider the polynomial $\frac{1}{k}(1+x)^k - \frac{1}{k}$ in $x\mathbb{Q}[x]$. Following [BDM], for $k \in \mathbb{Z}_+$, we define $a_j \in \mathbb{Q}$ for $j \in \mathbb{Z}_+$, by

$$\exp\left(-\sum_{j \in \mathbb{Z}_+} a_j x^{j+1} \frac{\partial}{\partial x}\right) \cdot x = \frac{1}{k}(1+x)^k - \frac{1}{k}. \quad (8)$$

For example, $a_1 = (1-k)/2$ and $a_2 = (k^2-1)/12$. Let $V = (V, Y, \mathbf{1}, \omega)$ be a VOSA. In $(\text{End } V)[[x^{1/2k}, x^{-1/2k}]]$, define

$$\Delta_k(x) = \exp\left(\sum_{j \in \mathbb{Z}_+} a_j x^{-\frac{j}{k}} L(j)\right) (k^{\frac{1}{2}})^{-2L(0)} \left(x^{\frac{1}{2k}(k-1)}\right)^{-2L(0)}. \quad (9)$$

For $v \in V$, and k any positive integer, denote by $v^j \in V^{\otimes k}$, for $j = 1, \dots, k$, the vector whose j -th tensor factor is v and whose other tensor factors are $\mathbf{1}$. Then for $g = (1\ 2\ \dots\ k)$, we have $gv^j = v^{j-1}$ for $j = 1, \dots, k$ where 0 is understood to be k .

Let (M, Y_M) be a V -module, and (M_σ, Y_σ) a σ -twisted V -module, where σ is the parity map on V . We define the g -twisted vertex operators for $V^{\otimes k}$ on M , for k odd, and on M_σ , for k even, as follows: Set

$$Y_g(v^1, x) = \begin{cases} Y_M(\Delta_k(x)v, x^{1/k}) & \text{for } k \text{ odd} \\ Y_\sigma(\Delta_k(x)v, x^{1/k}) & \text{for } k \text{ even} \end{cases} \quad (10)$$

and for $j = 0, \dots, k-1$, define

$$Y_g(v^{j+1}, x) = \lim_{x^{1/k} \rightarrow \eta^j x^{1/k}} Y_g(v^1, x). \quad (11)$$

Let V be an arbitrary VOSA and h an automorphism of V of finite order. Denote the categories of weak, weak admissible and ordinary h -twisted V -modules by $\mathcal{C}_w^h(V)$, $\mathcal{C}_a^h(V)$ and $\mathcal{C}^h(V)$, respectively. If $h = 1$, we habitually remove the index h .

Now again consider the VOSA, $V^{\otimes k}$, and the k -cycle $g = (1 \ 2 \ \dots \ k)$. For k odd, define

$$T_g^k : \mathcal{C}_w(V) \longrightarrow \mathcal{C}_w^g(V^{\otimes k}), \quad (M, Y_M) \mapsto (T_g^k(M), Y_g) = (M, Y_g). \quad (12)$$

For k even, define

$$T_g^k : \mathcal{C}_w^\sigma(V) \longrightarrow \mathcal{C}_w^g(V^{\otimes k}), \quad (M_\sigma, Y_\sigma) \mapsto (T_g^k(M_\sigma), Y_g) = (M_\sigma, Y_g). \quad (13)$$

The following theorem is proved in [B6] for k odd, and in [BV2] for k even.

Theorem 1. ([B6], [BV2])

(1) For k odd, the functor T_g^k is an isomorphism from the category $\mathcal{C}_w(V)$ of weak V -modules to the category $\mathcal{C}_w^g(V^{\otimes k})$ of weak $g = (1 \ 2 \ \dots \ k)$ -twisted $V^{\otimes k}$ -modules.

(2) For k even, the functor T_g^k is an isomorphism from the category $\mathcal{C}_w^\sigma(V)$ of weak parity-twisted V -modules to the category $\mathcal{C}_w^g(V^{\otimes k})$ of weak $g = (1 \ 2 \ \dots \ k)$ -twisted $V^{\otimes k}$ -modules.

(3) For any $k \in \mathbb{Z}_+$, the functor T_g^k restricted to the respective subcategories of weak admissible, ordinary or irreducible modules in $\mathcal{C}_w(V)$ or $\mathcal{C}_w^\sigma(V)$, respectively, is an isomorphism between these subcategories and the corresponding subcategory of weak admissible, ordinary or irreducible g -twisted $V^{\otimes k}$ -modules.

4 N=2 supersymmetric VOSAs, Ramond sectors, and mirror-twisted sectors

In this section, we recall the notions of N=1 or N=2 supersymmetric VOSA, following the notation and terminology of, for instance, [B1], [B2] and [B3]. First we will need the notion of several superextensions of the Virasoro algebra.

The $N=1$ Neveu-Schwarz algebra or $N=1$ superconformal algebra is the Lie superalgebra with basis consisting of the central element d , even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z} + \frac{1}{2}$, and supercommutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}d, \quad (14)$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}, \quad [G_r, G_s] = 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}d, \quad (15)$$

for $m, n \in \mathbb{Z}$, and $r, s \in \mathbb{Z} + \frac{1}{2}$. The $N=1$ Ramond algebra is the Lie superalgebra with basis the central element d , even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z}$, and supercommutation relations given by (14)–(15), where now $r, s \in \mathbb{Z}$.

The $N=2$ Neveu-Schwarz Lie superalgebra or $N=2$ superconformal algebra is the Lie superalgebra with basis consisting of the central element d , even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $j = 1, 2$ and $r \in \mathbb{Z} + \frac{1}{2}$, and such that the supercommutation relations are given as follows: L_n, d and $G_r^{(j)}$ satisfy the supercommutation relations for the $N=1$ Neveu-Schwarz algebra (14)–(15) for both $G_r = G_r^{(1)}$ and for $G_r = G_r^{(2)}$; the remaining relations are given by

$$[L_m, J_n] = -nJ_{m+n}, \quad [J_m, J_n] = \frac{1}{3}m\delta_{m+n,0}d \quad (16)$$

$$[J_m, G_r^{(1)}] = -iG_{m+r}^{(2)}, \quad [J_m, G_r^{(2)}] = iG_{m+r}^{(1)}, \quad [G_r^{(1)}, G_s^{(2)}] = i(s-r)J_{r+s}. \quad (17)$$

The $N=2$ Ramond algebra is the Lie superalgebra with basis consisting of the central element d , even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $r \in \mathbb{Z}$ and $j = 1, 2$, and supercommutation relations given by those of the $N=2$ Neveu-Schwarz algebra but with $r, s \in \mathbb{Z}$, instead of $r, s \in \mathbb{Z} + \frac{1}{2}$.

Note that there is an automorphism of the $N=2$ Neveu-Schwarz algebra given by

$$\kappa: G_r^{(1)} \mapsto G_r^{(1)}, \quad G_r^{(2)} \mapsto -G_r^{(2)}, \quad J_n \mapsto -J_n, \quad L_n \mapsto L_n, \quad d \mapsto d, \quad (18)$$

called the *mirror map* automorphism of the $N=2$ Neveu-Schwarz algebra.

Let $(V, Y, \mathbf{1}, \omega)$ be a VOSA, and suppose there exists $\tau \in V_{3/2}$ such that writing $Y(\tau, z) = \sum_{n \in \mathbb{Z}} \tau_n x^{-n-1} = \sum_{n \in \mathbb{Z}} G(n+1/2)x^{-n-2}$, the $G(n+1/2) = \tau_{n+1} \in (\text{End } V)^{(1)}$ generate a representation of the $N=1$ Neveu-Schwarz Lie superalgebra such that the $L(n)$ are the modes of ω . Then we call $(V, Y, \mathbf{1}, \tau)$ an $N=1$ Neveu-Schwarz VOSA, or an $N=1$ supersymmetric VOSA, or just an $N=1$ VOSA for short.

Suppose a VOSA, V , has two vectors $\tau^{(1)}$ and $\tau^{(2)}$ such that $(V, Y, \mathbf{1}, \tau^{(j)})$ is an $N=1$ VOSA for both $j = 1$ and $j = 2$, and the $\tau_{n+1}^{(j)} = G^{(j)}(n+1/2)$ generate a representation of the $N=2$ Neveu-Schwarz Lie superalgebra. Then we call such a VOSA an $N=2$ Neveu-Schwarz VOSA or an $N=2$ supersymmetric VOSA, or for short, an $N=2$ VOSA.

For the case of the parity map, σ , a σ -twisted V -module, for V an $N=1$ or $N=2$ VOSA, is naturally a representation of the $N=1$ or $N=2$ Ramond algebra, respectively. (See for instance [B4], [B5], as well as references therein).

Suppose V is an $N=2$ VOSA such that V has an automorphism g_κ which is a lift of the mirror map κ for the $N=2$ Neveu-Schwarz algebra. That is letting g_κ act by conjugation on $\text{End } V$, then g_κ restricts to the mirror map κ on the elements $L(n)$,

$J(n)$, and $G^{(j)}(r)$, for $n \in \mathbb{Z}$, $j = 1, 2$, and $r \in \mathbb{Z} + \frac{1}{2}$, which give the N=2 Neveu-Schwarz algebra representation on the N=2 VOSA, V . Following [B4], [B5], we call such an automorphism g_κ of an N=2 VOSA, V , a *mirror map*. Then a g_κ -twisted V -module is naturally a representation of the “mirror-twisted N=2 Neveu-Schwarz algebra”. The *mirror-twisted N=2 Neveu-Schwarz algebra* is the Lie superalgebra with basis consisting of even elements L_n , and J_r and central element d , odd elements $G_r^{(1)}$ and $G_n^{(2)}$, for $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, and supercommutation relations given as follows: The L_n and $G_r^{(1)}$ satisfy the supercommutation relations for the N=1 Neveu-Schwarz algebra with central charge d ; the L_n and $G_n^{(2)}$ satisfy the supercommutation relations for the N=1 Ramond algebra with central charge d ; and the remaining supercommutation relations are

$$[L_n, J_r] = -rJ_{n+r}, \quad [J_r, J_s] = \frac{1}{3}r\delta_{r+s,0}d, \quad [G_r^{(1)}, G_n^{(2)}] = -i(r-n)J_{r+n} \quad (19)$$

$$[J_r, G_s^{(1)}] = -iG_{r+s}^{(2)}, \quad [J_r, G_n^{(2)}] = iG_{r+n}^{(1)}. \quad (20)$$

Note that this mirror-twisted N=2 Neveu-Schwarz algebra is not isomorphic to the ordinary N=2 Neveu-Schwarz algebra [SS].

5 Mirror-twisted modules for the class of N=2 VOSAs of the form $V \otimes V$

There are large classes of N=2 VOSAs of the form $V \otimes V$ such that V is an N=1 VOSA, and $\tilde{\kappa} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, the signed transposition map given by Eqn. (1), is a mirror map for $V \otimes V$. Examples of such N=2 VOSAs, were studied in [B4]. These include the following examples: Let V_L be a rank d positive definite integral lattice VOSA or the d free boson vertex operator algebra, and let V_{fer}^d be the d free fermion VOSA. As noted in [B4], the VOSA $V = V_L \otimes V_{fer}^d$, is naturally an N=1 VOSA, and $V \otimes V$ is naturally an N=2 VOSA. This uses the construction of a VOSA from a positive definite integral lattice, following for instance [DL], [X], [BV1]. Such N=2 VOSAs have more than one mirror map as was shown in [B4], where the author constructed mirror-twisted modules for these VOSAs for the other mirror map.

For such N=2 VOSAs of the form $V \otimes V$, and for the signed transposition mirror-map $\tilde{\kappa}$, we have the following immediate corollary to Theorem 1.

Corollary 1. *The category of weak mirror-twisted $(V \otimes V)$ -modules for the signed transposition mirror map automorphism of an N=2 VOSA of the form $V \otimes V$ is isomorphic to the category of weak N=1 Ramond-twisted V -modules (i.e., parity-twisted V -modules). In addition, the subcategories of weak admissible, ordinary, or irreducible modules are isomorphic.*

In particular, it follows that if M_σ is a representation of the N=1 Ramond algebra such that M_σ is a weak parity-twisted module for an N=1 VOSA, V , and such that

$V \otimes V$ is an N=2 VOSA, then M_σ is also naturally a representation of the mirror-twisted N=2 superconformal algebra and is a weak $\tilde{\kappa}$ -twisted module for $V \otimes V$.

Furthermore, from the construction of such modules given by the functor T_g^k for $k = 2$ as in (10), (11), (13), (see also [BV2]), we have as a consequence of Corollary 6.5 in [BV2], the following:

Corollary 2. $M_{\tilde{\kappa}}$ is an ordinary $\tilde{\kappa}$ -twisted $(V \otimes V)$ -module with graded dimension

$$\dim_q M_{\tilde{\kappa}} = \text{tr}_{M_{\tilde{\kappa}}} q^{-2c/24 + L^{\tilde{\kappa}}(0)} = q^{-c/12} \sum_{\lambda \in \mathbb{C}} \dim(M_\lambda) q^\lambda$$

if and only if $(T_{\tilde{\kappa}}^2)^{-1}(M_{\tilde{\kappa}}) = M_{\tilde{\sigma}}$ is an ordinary σ -twisted V -module with graded dimension

$$\dim_q (T_{\tilde{\kappa}}^2)^{-1}(M_{\tilde{\kappa}}) = \text{tr}_{M_{\tilde{\kappa}}} q^{-c/24 + L^\sigma(0)} = \dim_{q^2} M_{\tilde{\kappa}},$$

where c is the central charge of V .

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